Paper-V

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In last lecture, we have discussed the idea of vector integration. Line integral, Surface integral and Volume integral. We have also discussed a problem on line integral. Here I consider the Surface integral and discuss how to evaluate it for a given problem.

- Surface integral expression $\rightarrow \int_S \vec{V} \cdot d\vec{\sigma}$

$S \leftarrow$ integrated over surface S

$\vec{V} \rightarrow$ vector function

$d\vec{\sigma} = \hat{n}\, dA$, $\hat{n} \rightarrow$ unit normal to surface

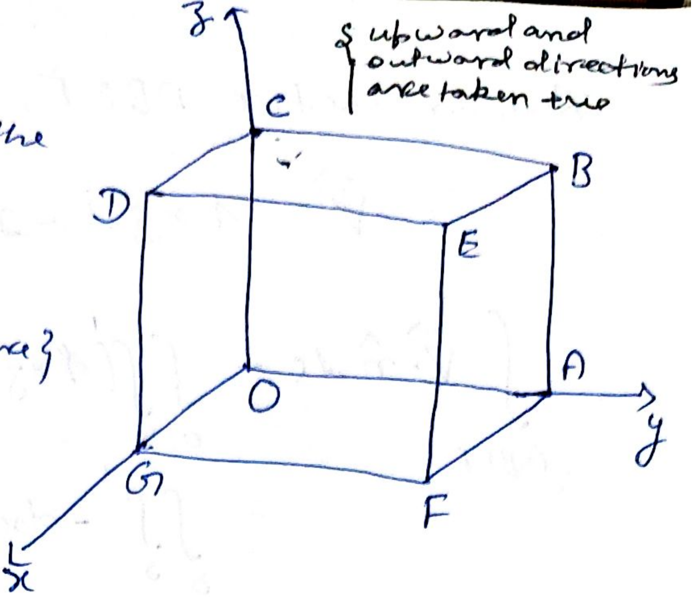
~~Q: If $\vec{V} = \hat{i}\, 4xz \ldots y^2 \ldots \hat{k}\, yz$~~

Q. If $\vec{V} = 4xz\, i - y^2 j + yz\, k$, then evaluate

$\int_S \vec{V} \cdot \hat{n}\, ds$, where S is the surface of the Cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

Soln.

{ upward and outward directions are taken true



We consider the face DEFG of the cube:

here $\hat{n} = i$, $x = 1$

$\therefore \vec{V} = 4z i - y^2 j + yz\hat{k}$ {for $x=1$ here}

$$\int_{DEFG} \vec{V}\cdot\hat{n}\, ds$$

$$= \int_0^1\int_0^1 (4z i - y^2 j + yz k)\cdot i \, dy\, dz$$

{ $ds = dy\,dz$ for DEFG, y & z limits are from 0 to 1

$$= \int_0^1 \{dy\}$$

$$= \int_0^1\int_0^1 4z \, dy\, dz = \int_0^1 4z\, dy = [2z^2]_0^1 = 2$$

$$\int_{DEFG} \vec{V}\cdot\hat{n}\, ds = 2 \quad \text{———} \quad (1)$$

Again, For the face ABCO, $\hat{n} = -i$, $x = 0$

$$\vec{V} = -y^2 j + yz\hat{k}$$

{ Put $x=0$ in $\vec{V} = 4xz\hat{i} - y^2 j + yz k$ for this case and $ds = dy\,dz$.

$$\int_{ABCO} \vec{V}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (-y^2 j + yz k)\cdot(-i)\, dy\, dz = 0$$

or $$\int_{ABCO} \vec{V}\cdot\hat{n}\, ds = 0 \quad \text{———} \quad (2)$$

Next, for the face ABEF, $\hat{n} = j,\ y = 1$

$$\vec{V} = 4xz\,i - j + z\,k$$

$$\int_{ABEF} \vec{V}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\,i - j + z\,k)\cdot j\,dx\,dy$$

$$= \int_0^1\int_0^1 -\,dx\,dz = -\int_0^1 dz = -1.$$

$$\int_{ABEF} \vec{V}\cdot\hat{n}\,ds = -1 \quad \text{—— (3)}$$

~~Again, for face ABEF, we have~~ $\cancel{\hat{n} = j,\ y=1}$

$$\cancel{\vec{V} = 4xz\,i - j}$$

For the face OGDC, we have $\hat{n} = -j,\ y = 0$

$$\vec{V} = 4xz\,i$$

$$\therefore \int_{OGDC} \vec{V}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\,i)\cdot(-j)\,dx\,dz = 0$$

$$\text{or} \quad \int_{OGDC} \vec{V}\cdot\hat{n}\,ds = 0 \quad \text{—— (4)}$$

For the face BCDE, we have $\hat{n} = k,\ z = 1$

$$\therefore \vec{V} = 4x\,i - y^2\,j + y\,k$$

$$\int_{BCDE} \vec{V}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4x\,i - y^2 j + y\,k)\cdot k\,dx\,dy$$

$$= \int_0^1\int_0^1 y\,dx\,dx = 1\int_0^1 y\,dy = \frac{1}{2}$$

$$\therefore \int_{BCDE} \vec{V}\cdot\hat{n}\,ds = \frac{1}{2} \quad \text{--- (5)}$$

Again for the face, AFGO, we have $\hat{n} = -k$, $z = 0$.

$$\vec{V} = -y^2 \hat{j}.$$

$$\int_{AFGO} \vec{V}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (-y^2 \hat{j})(-\hat{k})\,dx\,dy = 0 \quad \text{--- (6)}$$

$$\because \int_S \vec{V}\cdot\hat{n}\,ds = \int_{DEFG} \vec{V}\cdot\hat{n}\,ds + \int_{ABCO} \vec{V}\cdot\hat{n}\,ds + \int_{ABFE} \vec{V}\cdot\hat{n}\,ds$$

$$+ \int_{OGDG} \vec{V}\cdot\hat{n}\,ds + \int_{BCDE} \vec{V}\cdot\hat{n}\,ds + \int_{AFGO} \vec{V}\cdot\hat{n}\,ds.$$

using Equation (1), (2), (3), (4), (5), and (6), we can write

$$\int_S \vec{V}\cdot\hat{n}\,ds = 2 + 0 - 1 + 0 + \frac{1}{2} + 0 = \frac{3}{2}$$

$$\boxed{\int_S \vec{V}\cdot\hat{n}\,ds = \frac{3}{2}}$$

H.W. Calculate surface integral of ~~$\vec{V} = 2xz\hat{i} + (x+2)\hat{j} + y(z^2-3)$~~

$\vec{V} = 2xz\hat{i} + (x+2)\hat{j} + y(z^2-3)\hat{k}$ over five sides (excluding bottom) of the cubical box. Assume the upward and outward directions to be +ve. {See Fig. of the question}

The bottom side of the fig. is AFGO.